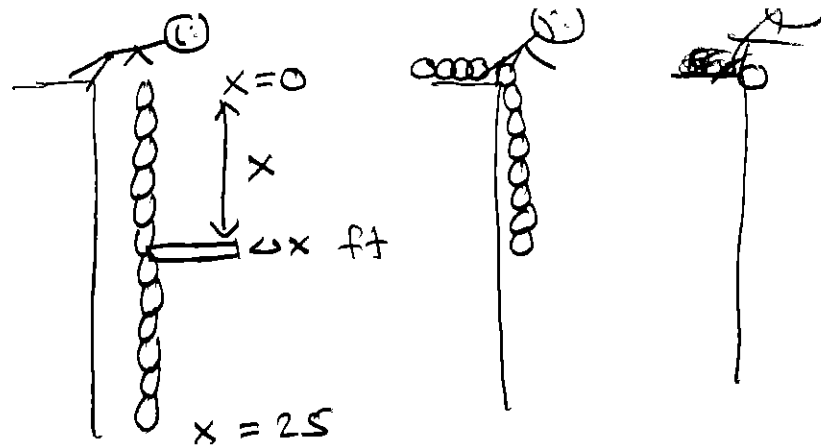


Examples:

1. (Chains/Cables) You are lifting a heavy chain to the top of a building. The chain has a density of 3 lbs/foot. The chain hangs over the side by 25 feet before you start pulling it up. How much work is done in pulling the chain all the way to the top?

$$\begin{aligned} \text{WORK} &= \int_0^{25} \overbrace{x}^{\text{DIST}} \cdot \overbrace{3 dx}^{\text{FORCE}} \\ &= \frac{1}{2} x^2 \cdot 3 \Big|_0^{25} \\ &= \frac{3}{2} (25)^2 - (0)^2 \\ &= \frac{3}{2} \cdot 625 \\ &= \boxed{937.5 \text{ ft-lbs}} \end{aligned}$$

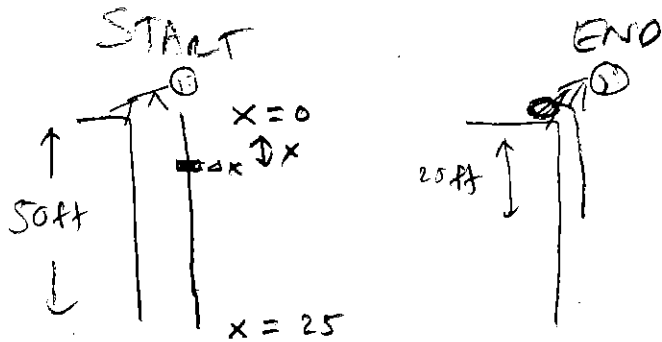


$$\begin{aligned} \text{WEIGHT OF SUBDIVISION} &= 3 \frac{\text{lbs}}{\text{ft}} \cdot \Delta x \text{ ft} = 3 \Delta x \\ \text{DIST} &= x \end{aligned}$$

Example:

A 50 foot cable with density 4 lbs/ft is hanging over the side of a tall building.

Find the total work done in lifting the cable half way up.



For $0 \leq x \leq 25$:

$$\text{FORCE} = 40x$$

$$\text{DIST} = x$$

$$\int_0^{25} x \cdot 4 dx = 2x^2 \Big|_0^{25} \\ = 1250 \text{ ft-lbs}$$

For $25 \leq x \leq 50$:

$$\text{FORCE} = 40x$$

$$\text{DIST} = 25$$

$$\int_{25}^{50} 25 \cdot 4 dx = 100x \Big|_{25}^{50} \\ = 100(50 - 25) \\ = 2500 \text{ ft-lbs}$$

$$\text{TOTAL} = 1250 + 2500 = \boxed{3750 \text{ ft-lbs}}$$

Example: (You do – like HW) }

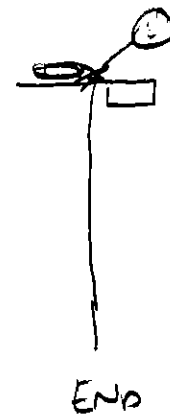
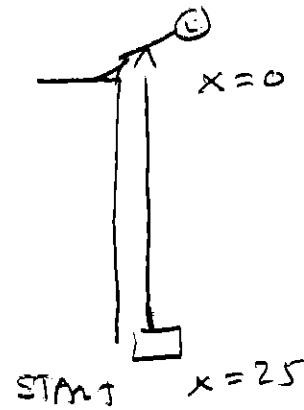
A cable with density 3 lbs/ft is being used to lift a 50 pound weight from the ground to the top of a 25 foot building. Find the total work done.

Step 1: Draw a picture.

Step 2: Break up the problem:

- (a) Work to lift the 50 lbs weight?
- (b) Work to lift the cable?

Step 3: Add these together.



OBJECT

$$F = 50 \text{ lbs}$$

$$D = \Delta x$$

$$\int_0^{25} 50 \, dx$$

$$50x \Big|_0^{25}$$

$$50 \cdot (25 - 0)$$

$$1250 \text{ ft-lbs}$$

CABLE

$$F = 3 \Delta x$$

$$D = x$$

$$\int_0^{25} x \cdot 3 \, dx$$

$$\frac{3}{2} x^2 \Big|_0^{25}$$

$$\frac{3}{2} ((25)^2 - 0)$$

$$937.5 \text{ ft-lbs}$$

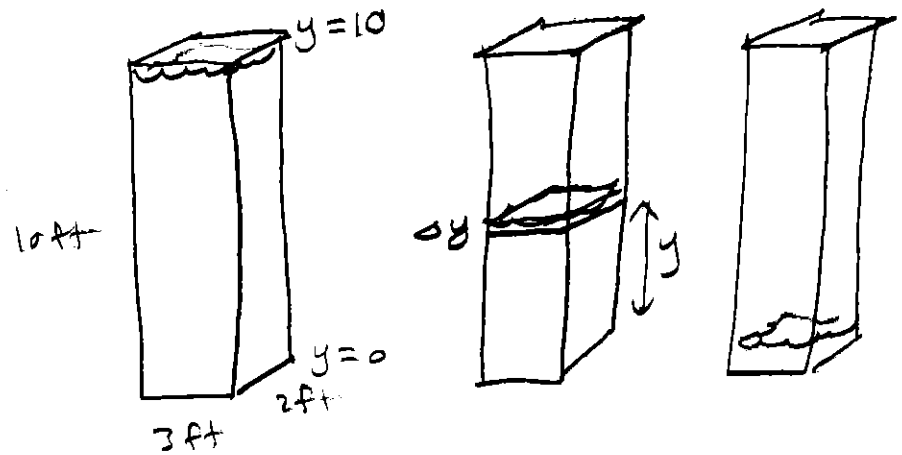
TOTAL

$$\int_0^{25} 50 + 3x \, dx = 1250 + 937.5$$

$$= 2187.5 \text{ ft-lbs}$$

2. (Pumping Liquid) You are pumping water out of a tank. The tank is a rectangular box with a base of 2 ft by 3 ft and height of 10ft. The density of water is 62.5 lbs/ft³.

If the tank starts full, how much work is done in pumping all the water to the top and out over the side?



$$\text{Work} = \int_0^{10} \overbrace{(10-y)}^{\text{DIST}} \cdot \overbrace{62.5 \cdot 6 \, dy}^{\text{FORCE}}$$

\uparrow
 $\frac{\text{lbs}}{\text{ft}^3}$ ft^3

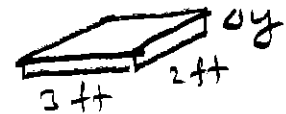
$$= 375 \int_0^{10} 10 - y \, dy$$

$$= 375 (10y - \frac{1}{2}y^2 \Big|_0^{10})$$

$$= 375 (100 - 50) = 375 \cdot 50 = \boxed{18,750 \text{ ft-lbs}}$$

$$\begin{aligned} \text{WEIGHT} &= 62.5 \frac{\text{lbs}}{\text{ft}^3} \cdot 3 \cdot 2 \cdot dy \text{ ft}^3 \\ &= 62.5 \cdot 6 \, dy \end{aligned}$$

$$\begin{aligned} \text{DIST LIFTED} &= 10 - y \end{aligned}$$

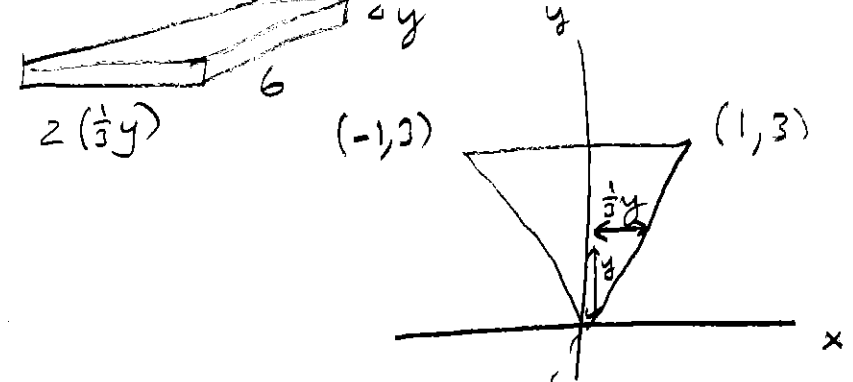
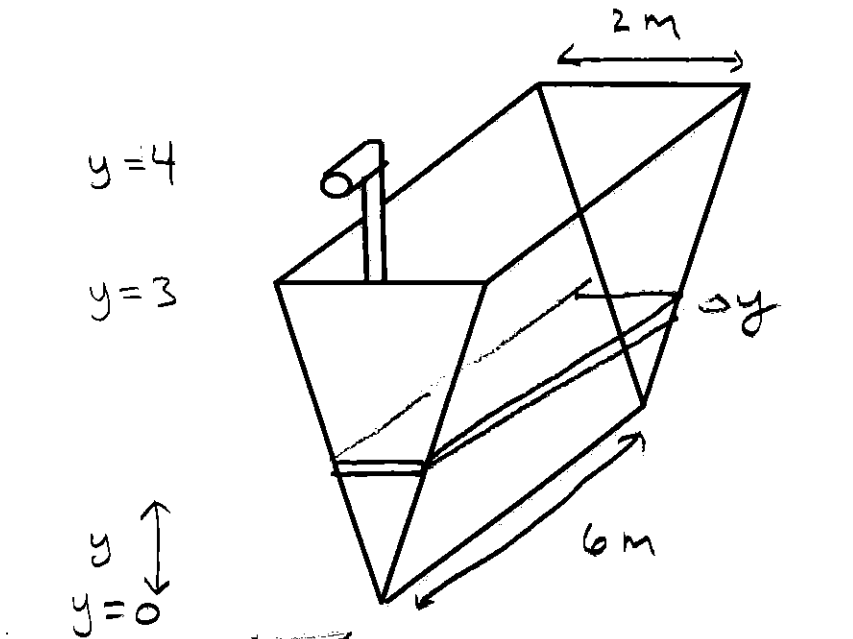


Example:

Consider the tank shown at right.

The height is 3 meters, the width at the top is 2 meters and the length is 6 meters. Also we are pumping the water up to 4 meters above the ground (1 meter above the top edge).

If it starts full, how much work is done to pump it all out?



$$\text{slope} = \frac{3-0}{1-0} = 3$$

$$y = 3x$$

$$x = \frac{1}{3}y$$

$$\begin{aligned} \text{WEIGHT OF} &= 9800 \frac{\text{N}}{\text{m}^3} \cdot \frac{2}{3}y \cdot 6 \cdot dy \text{ m}^3 \\ \text{HORIZ. SLICE} &= 9800 \cdot 4y \cdot dy \end{aligned}$$

$$\text{DIST LIFTED} = 4 - y$$

$$\int_0^3 \overbrace{(4-y)}^{\text{DIST}} \cdot \overbrace{9800 \cdot 4y \cdot dy}^{\text{FORCE}} \text{ m}^3$$

$$= 39200 \int_0^3 (4-y)y \, dy = 39200 \int_0^3 (4y - y^2) \, dy$$

$$= 39200 \left(2y^2 - \frac{1}{3}y^3 \right) \Big|_0^3 = 39200 \left((18 - 9) - 0 \right) = \boxed{352800 \text{ Joules}}$$

Quick Summary:

$$\begin{aligned}\text{Work} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n (\text{FORCE})(\text{DIST}) \\ &= \int_a^b (\text{FORCE})(\text{DIST})\end{aligned}$$

Problem type 1: (Leaky bucket/spring)

Leaking at constant rate $\rightarrow f(x) = mx+b$

Spring (Hooke's Law) $\rightarrow f(x) = kx$

Force given $\rightarrow f(x) = \text{force}$

FORCE = $f(x_i)$, DISTANCE = Δx

$$\text{WORK} = \int_a^b f(x) dx$$

Problem type 2: (Chain/pumping)

FORCE = weight of a horizontal slice

DIST = distance moved by that slice

Chain:

k = density = force per distance

FORCE = weight of slice = $k\Delta x$

DIST = distance moved by slice

(typically x if you label like me)

$$\text{WORK} = \int_0^b x k dx$$

Pumping:

Density of water

$$= 1000 \text{ kg/m}^3 = 9800 \text{ N/m}^3$$

$$= 62.5 \text{ lbs/ft}^3$$

k = density = weight per volume

FORCE = $k \text{ vol} = k(\text{hor. slice area})\Delta y$

DIST = distance moved by slice

(typically $a-y$ if you label like me)

$$\text{WORK} = \int_0^b (a-y)k(\text{slice area})dy$$

6.5 Average Value

The average value of the n numbers:

$$y_1, y_2, y_3, \dots, y_n$$

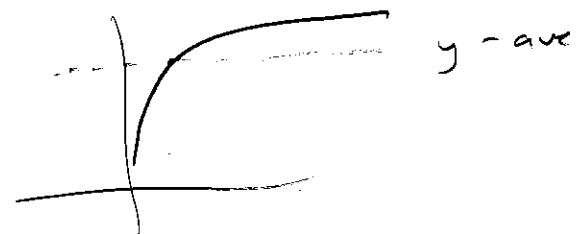
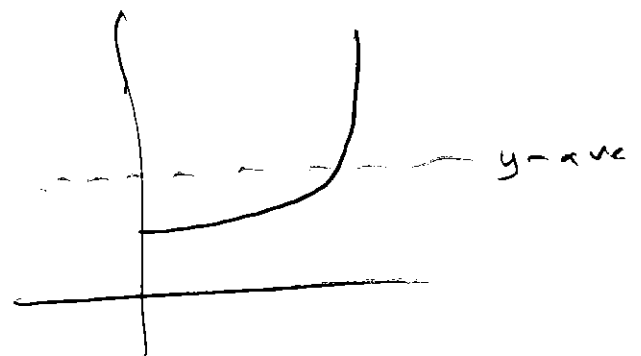
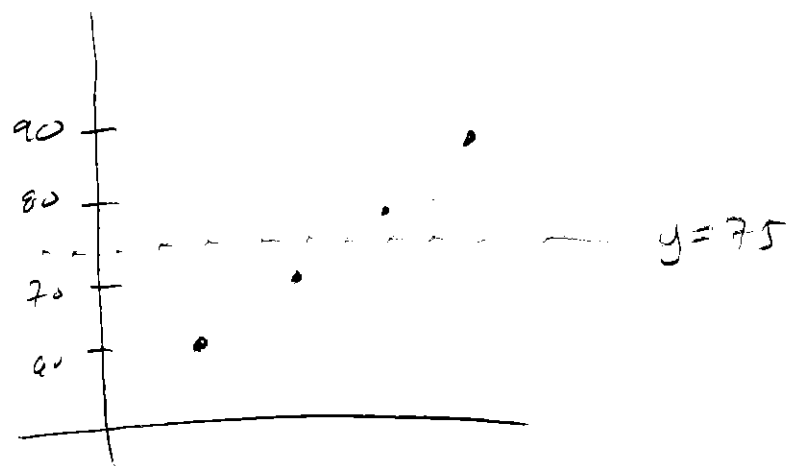
is given by

$$\frac{y_1 + y_2 + y_3 + \dots + y_n}{n} = y_1 \frac{1}{n} + \dots + y_n \frac{1}{n}$$

Goal: We want the average value of **all** the y -values of some function $y = f(x)$ over an interval $x = a$ to $x = b$.

Ex | 4 TEST SCORE
60, 70, 80, 90

$$\text{AVE TEST SCORE} = \frac{60 + 70 + 80 + 90}{4} = 75$$



Derivation:

1. Break into n equal subdivisions

$$\Delta x = \frac{b-a}{n}, \text{ which means } \frac{\Delta x}{b-a} = \frac{1}{n}$$

2. Compute y -value at each tick mark

$$y_1 = f(x_1), y_2 = f(x_2), \dots, y_n = f(x_n)$$

3. Ave $\approx f(x_1) \frac{\Delta x}{b-a} + \dots + f(x_n) \frac{\Delta x}{b-a}$

$$\text{Average} \approx \frac{1}{b-a} \sum_{i=1}^n f(x_i) \Delta x$$

Thus, we can define

$$\text{Average} = \frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

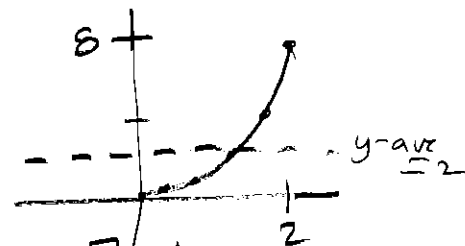
Which means the exact average y -value of $y = f(x)$ over $x = a$ to $x = b$ is

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx$$

Ex) $f(x) = x^3$, $x=0$ to $x=2$

$$\Delta x = \frac{2-0}{5} = \frac{2}{5}$$

$$x_0, x_1, x_2, x_3, x_4, x_5, 2$$



$$\left[\left(\frac{2}{5}\right)^3 + \left(\frac{4}{5}\right)^3 + \left(\frac{6}{5}\right)^3 + \left(\frac{8}{5}\right)^3 + (2)^3 \right] \frac{1}{5}$$

$$\Delta x = \frac{2}{5} \Rightarrow \frac{\Delta x}{2-0} = \frac{1}{5}$$

$$\frac{1}{2} \left[\left(\frac{2}{5}\right)^3 \Delta x + \left(\frac{4}{5}\right)^3 \Delta x + \dots + \left(\frac{8}{5}\right)^3 \Delta x + (2)^3 \Delta x \right]$$

$$\frac{1}{b-a} \int_a^b f(x) dx$$

$$\frac{1}{2-0} \int_0^2 x^3 dx$$

$$\frac{1}{2} \left(\frac{1}{4} x^4 \Big|_0^2 \right)$$

$$\frac{1}{8} \left((2^4) - (0^4) \right) = \frac{16}{8} = 2$$

$y\text{-ave} = 2$, it occurs when $x = 2^{1/2}$